

# AFFINE SPHERES WITH PRESCRIBED BLASCHKE METRIC

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**ABSTRACT.** It is proved that the equality  $\Delta \ln |\kappa - \lambda| = 6\kappa$ , where  $\kappa$  is the Gaussian curvature of a metric tensor  $g$  on a 2-dimensional manifold is a sufficient and necessary condition for local realizability of the metric as the Blaschke metric of some affine sphere. Consequently, the set of all improper local affine spheres with nowhere-vanishing Pick invariant can be parametrized by harmonic functions.

## 1. INTRODUCTION

Affine spheres are still a mysterious class, even in the 2-dimensional case. The following fact has been known since Blaschke's times, [1]. For an affine sphere in  $\mathbf{R}^3$  whose Pick invariant is nowhere vanishing the following equality is satisfied

$$(1) \quad \Delta \ln |J| = 6\kappa,$$

where  $\kappa$  is the Gaussian curvature of the Blaschke metric and  $J$  is the Pick invariant. The Laplacian is taken relative to the Blaschke metric. The affine theorem egregium says that  $J = \kappa - \rho$ , where  $\rho$  is the affine scalar curvature. For an affine sphere with the shape operator  $S = \lambda \text{id}$  the scalar curvature is equal to  $\lambda$ . Therefore, the equality (1) can be written as  $\Delta \ln |\kappa - \lambda| = 6\kappa$ . It has not been noticed, however, that this equality is also a sufficient condition for local realizability of a metric as the Blaschke metric on an affine sphere in  $\mathbf{R}^3$ . Realizability of prescribed objects on submanifolds belongs to fundamental problems in all types of geometry of submanifolds. The main aim of this paper is to prove the following result

**Theorem 1.1.** *Let  $g$  be a metric tensor field on a 2-dimensional manifold  $M$ . It can be locally realized as the Blaschke metric on an affine sphere with nowhere-vanishing Pick invariant if and only if*

$$(2) \quad \Delta \ln |\kappa - \lambda| = 6\kappa$$

*for some constant  $\lambda$  such that  $\kappa - \lambda \neq 0$  everywhere.*

Note that the 2-dimensional affine sphere with vanishing Pick invariant are classified. Namely, by the affine theorem egregium we know that if an affine sphere has vanishing Pick invariant then the Gaussian curvature of the Blaschke metric is constant. A complete classification of such affine spheres in  $\mathbf{R}^3$  is given, for instance, in Section 5 of Chapter III in [2].

As corollaries of Theorem 1.1 we shall prove

**Corollary 1.2.** *An improper locally strongly convex affine sphere in  $\mathbf{R}^3$  with nowhere-vanishing Pick invariant is analytic.*

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Note that not all improper affine spheres are analytic. For instance, the surface given by the equation  $z = xy + \Phi(x)$ , where  $\Phi$  a smooth but non-analytic function is an affine sphere. This sphere has vanishing Pick invariant and its Blaschke metric is indefinite, see [2].

In the following corollary  $g_0$  stands for the canonical metric tensor field in  $\mathbf{R}^2$  and the harmonicity is relative to  $g_0$ .

**Corollary 1.3.** *Let  $\bar{g} = e^L g_0$  be a metric tensor of constant Gaussian curvature  $\pm 2$  defined in a neighborhood  $U$  of  $p \in \mathbf{R}^2$ . Let  $h$  be a harmonic function defined on  $U$ . Then  $g = e^{\frac{h-L}{2}} g_0$  can be locally realized as the Blaschke metric in a neighborhood of  $p$  on some improper affine sphere.*

Since metric structures of the same constant Gaussian curvature are locally isometric, the above theorem says, roughly speaking, that local improper affine spheres with nowhere-vanishing Pick invariant can be parametrized by harmonic functions.

## 2. AFFINE SPHERES

Let  $f : M \rightarrow \mathbf{R}^{n+1}$  be a hypersurface in  $\mathbf{R}^{n+1}$ . At the beginning, for simplicity, we assume that  $M$  is connected and oriented. On  $\mathbf{R}^{n+1}$  we have the volume form given by the standard determinant. The standard flat connection on  $\mathbf{R}^{n+1}$  will be denoted by  $\tilde{\nabla}$ . Let  $\xi$  be a transversal vector field for  $f$  (consistent with the orientation on  $M$ ). The induced volume form on  $M$  is given by

$$(3) \quad \nu_\xi(X_1, \dots, X_n) = \det(f_*X_1, \dots, f_*X_n, \xi).$$

We have the following Gauss formula

$$(4) \quad \tilde{\nabla}_X(f_*Y) = f_*(\nabla_X Y) + g(X, Y)\xi,$$

for vector fields  $X, Y$  on  $M$ . It is known that  $\nabla$  is a torsion-free connection and  $g$  is a symmetric bilinear form on  $M$ . The connection  $\nabla$  is called the induced connection and  $g$  - the second fundamental form of  $f$ . The conformal class of  $g$  is independent of the choice of a transversal vector field  $\xi$ . A hypersurface is called non-degenerate if  $g$  is non-degenerate at each point of  $M$ .

From now on we shall consider only non-degenerate hypersurfaces. Hence the second fundamental form  $g$  on  $M$  is a metric tensor field (maybe indefinite). The induced volume form on  $M$  is, in general, different than the volume form  $\nu_g$  determined by  $g$ . If  $\nu_\xi = \nu_g$  we say that the apolarity condition is satisfied. A transversal vector field  $\xi$  is called equiaffine if  $\nabla \nu_\xi = 0$  on  $M$ . If a transversal vector field is equiaffine then  $\nabla g$  as a cubic form (that is,  $\nabla g(X, Y, Z) = (\nabla_Z g)(Y, Z)$ ) is totally symmetric.

The following theorem is central in the classical affine differential geometry, see [3],[2].

**Theorem 2.1.** *Let  $f : M \rightarrow \mathbf{R}^{n+1}$  be a non-degenerate hypersurface. There exists a unique equiaffine transversal vector field  $\xi$  such that  $\nu_\xi = \nu_g$ .*

The unique transversal vector field is called the affine normal vector field. If the affine lines determined by the affine normal vector field meet at one point or are parallel then the hypersurface is called an affine sphere - proper in the first case and improper in the second one.

Affine spheres are also described by the affine shape operator. Namely, let  $\xi$  be the affine normal vector field for a hypersurface immersion  $f$ . By differentiating  $\xi$  relative to  $\tilde{\nabla}$  we get

$$(5) \quad \tilde{\nabla}_X \xi = -f_*(SX)$$

for some  $(1,1)$ -tensor field  $S$  on  $M$ . The tensor field  $S$  is called the affine shape operator. The fact that  $f$  is an affine sphere is equivalent to the condition  $S = \lambda \text{id}$ , where  $\lambda$  is a real number, non-zero for a proper sphere and zero for an improper sphere.

We have the following fundamental theorem for equiaffine hypersurfaces, see [3], [2].

**Theorem 2.2.** *Let  $\nabla$  be a torsion-free connection on a simply connected manifold  $M$ ,  $g$  be a symmetric bilinear non-degenerate form and  $S$  is a  $(1,1)$ -tensor field on  $M$  such that the following fundamental equations are satisfied:*

$$(6) \quad R(X, Y)Z = g(Y, Z)SX - g(X, Z)SY \quad - \text{ Gauss},$$

$$(7) \quad g(X, SY) = g(SX, Y) \quad - \text{ Ricci},$$

$$(8) \quad \nabla g(X, Y, Z) = \nabla g(Y, X, Z) \quad - \text{ I Codazzi},$$

$$(9) \quad \nabla S(X, Y) = \nabla S(Y, X) \quad - \text{ II Codazzi}$$

for every  $X, Y, Z \in T_x M$ ,  $x \in M$ , where  $R$  is the curvature tensor for  $\nabla$ . There is an immersion  $f : M \rightarrow \mathbf{R}^{n+1}$  and its equiaffine transversal vector field  $\xi$  such that  $\nabla$ ,  $g$ ,  $S$  are the induced connection, the second fundamental form and the shape operator for the immersion  $f$  equipped with the transversal vector field  $\xi$ . The immersion is unique up to an equiaffine transformation of  $\mathbf{R}^{n+1}$ . If moreover  $\nabla \nu_g = 0$  then  $\xi$  is the affine normal (up to a constant) for  $f$ .

For an affine sphere with  $S = \lambda \text{id}$  the fundamental equations reduce to the two equations

$$(10) \quad \begin{aligned} R(X, Y)Z &= \lambda \{g(Y, Z)X - g(X, Z)Y\}, \\ (\nabla g)(X, Y, Z) &= (\nabla g)(Y, X, Z). \end{aligned}$$

As a consequence of the fundamental theorem we have

**Corollary 2.3.** *Let  $M$  be a manifold equipped with a metric tensor field  $g$ , a torsion-free connection  $\nabla$  such that the equations (10) are satisfied for some constant real number  $\lambda$  and  $\nabla \nu_g = 0$ . For each point  $x$  of  $M$  there is a neighborhood  $U$  of  $x$  and an immersion  $f : U \rightarrow \mathbf{R}^{n+1}$  which is an affine sphere whose shape operator is equal to  $\lambda \text{id}$ .*

Proof. It is sufficient to define  $S = \lambda \text{id}$ . □

From now on we shall deal with the 2-dimensional case. For a connection on a 2-dimensional manifold the curvature tensor is determined by its Ricci tensor. Namely we have

$$(11) \quad R(X, Y)Z = \text{Ric}(Y, Z)X - \text{Ric}(X, Z)Y$$

for any vectors  $X, Y, Z \in T_x M$ ,  $x \in M$ . Therefore the Gauss equation for a 2-dimensional sphere is equivalent to the condition  $\text{Ric} = \lambda g$ . Hence we have

**Corollary 2.4.** *Let a metric tensor  $g$  and a torsion-free connection  $\nabla$  be given on a two-dimensional manifold  $M$ . They can be locally realized on an affine sphere if and only if the cubic form  $\nabla g$  is totally symmetric,  $\text{Ric} = \lambda g$  for some constant real number  $\lambda$  and  $\nabla \nu_g = 0$ .*

### 3. AFFINE CONNECTIONS, VOLUME FORMS AND THE RICCI TENSOR

All connections considered in this paper are torsion-free. For any connection  $\nabla$  and a metric tensor field  $g$  we denote by  $K$  the difference tensor, that is,  $K_X Y = \nabla_X Y - \hat{\nabla}_X Y$ , where  $\hat{\nabla}$  the Levi-Civita connection of  $g$ . Set  $K(X, Y) = K_X Y$ . Since both connections  $\nabla$  and  $\hat{\nabla}$  are without torsion,  $K$  is symmetric for  $X, Y$ . The cubic form  $\nabla g$  is symmetric if and only if  $K$  is symmetric relative to  $g$ , i.e.  $g(K(X, Y), Z) = g(K(X, Z), Y)$  for every  $X, Y, Z$ . Indeed, we have

$$(\nabla_X g)(Y, Z) = (K_X g)(Y, Z) = -g(K(X, Y), Z) - g(Y, K(X, Z)).$$

Assume that the cubic form  $\nabla g$  is symmetric. Since

$$(12) \quad 2\nabla_X \nu_g = \text{tr}_g(\nabla_X g)(\cdot, \cdot) \nu_g,$$

the condition  $\nabla \nu_g = 0$  is equivalent to the condition

$$(13) \quad \text{tr}_g(\nabla_X g)(\cdot, \cdot) = 0$$

for every  $X \in TM$ . Since

$$(14) \quad \nabla g(X, Y, Z) = -2g(K(X, Y), Z),$$

we have  $\nabla \nu_g = 0$  if and only if  $\text{tr}_g K_X = 0$  for every  $X$ .

Recall that the divergence of  $X \in \mathcal{X}(M)$  relative to a connection  $\nabla$  is defined as  $\text{div}^\nabla X = \text{tr} \{Y \rightarrow \nabla_Y X\}$ . Recall also that for a torsion-free connection  $\nabla$  there is (locally) a volume form  $\nu$  such that  $\nabla \nu = 0$  if and only if the Ricci tensor of  $\nabla$  is symmetric. A pair  $(\nabla, \nu)$  is then called an equiaffine structure. For a fixed coordinate system we have

$$\begin{aligned} (\nabla_{\partial_i} \nu)(\partial_1, \dots, \partial_n) &= \partial_i(\nu(\partial_1, \dots, \partial_n)) - \nu(\nabla_{\partial_i} \partial_1, \dots, \partial_n) - \dots - \nu(\partial_1, \dots, \nabla_{\partial_i} \partial_n) \\ &= \partial_i(\nu(\partial_1, \dots, \partial_n)) - \nu(\nabla_{\partial_1} \partial_i, \dots, \partial_n) - \dots - \nu(\partial_1, \dots, \nabla_{\partial_n} \partial_i) \\ &= \partial_i(\nu(\partial_1, \dots, \partial_n)) - (\text{div}^\nabla \partial_i) \nu(\partial_1, \dots, \partial_n). \end{aligned}$$

It follows that  $\nabla \nu = 0$  if and only if  $\partial_i(\ln(\nu(\partial_1, \dots, \partial_n))) = \text{div}^\nabla \partial_i$ . Assume now that  $\nu_g$  is a volume form determined by a metric tensor  $g$  (not necessarily positive definite). Then

$$\nu_g(\partial_1, \dots, \partial_n)^2 = G,$$

where  $G = |\det[g_{kl}]|$  and  $g_{kl} = g(\partial_k, \partial_l)$  for  $1 \leq k, l \leq n$ . Hence  $\nabla \nu_g = 0$  if and only if  $\text{div}^\nabla \partial_i = (\ln G)_i / 2$  for  $i = 1, \dots, n$ .

Let  $\Gamma_{ij}$  be the Christoffel symbols of a connection  $\nabla$ . In general, we have the following formula for the Ricci tensor  $\text{Ric}$  of  $\nabla$

$$(15) \quad \text{Ric}(\partial_i, \partial_j) = \sum_{k=1}^n (\Gamma_{ij}^k)_k - (D_j)_i + \Lambda_{ij}$$

where  $D_i = \text{div}^\nabla \partial_i$  and

$$(16) \quad \Lambda_{ij} = \sum_{k,l=1}^n [\Gamma_{kj}^l \Gamma_{il}^k - \Gamma_{ij}^l \Gamma_{kl}^k].$$

Since the connection is torsion-free,  $\Lambda_{ij}$  is symmetric for  $i, j$ . From now on we assume that  $M$  is 2-dimensional and  $\text{Ric}$  is symmetric. Let  $r_{ij}$  be the components of the Ricci tensor in a coordinate system. Then we have

$$(17) \quad \begin{aligned} \Lambda_{11} &= \Gamma_{11}^2 \Gamma_{12}^1 + (\Gamma_{12}^2)^2 - \Gamma_{11}^1 \Gamma_{12}^2 - \Gamma_{11}^2 \Gamma_{22}^2, \\ \Lambda_{12} &= \Gamma_{22}^1 \Gamma_{11}^2 - \Gamma_{12}^1 \Gamma_{12}^2, \\ \Lambda_{22} &= (\Gamma_{21}^1)^2 + \Gamma_{12}^2 \Gamma_{22}^1 - \Gamma_{22}^1 \Gamma_{11}^1 - \Gamma_{22}^2 \Gamma_{12}^1, \end{aligned}$$

$$(18) \quad \begin{aligned} (\Gamma_{12}^2)_1 &= (\Gamma_{11}^2)_2 - \Lambda_{11} - r_{11}, \\ (\Gamma_{12}^1)_1 &= -(\Gamma_{12}^2)_2 + \Lambda_{12} + (D_1)_2 + r_{12}, \\ (\Gamma_{22}^1)_1 &= -(\Gamma_{22}^2)_2 + \Lambda_{22} + (D_2)_2 + r_{22}, \end{aligned}$$

$$(19) \quad \begin{aligned} D_1 &= \Gamma_{11}^1 + \Gamma_{21}^2, \\ D_2 &= \Gamma_{12}^1 + \Gamma_{22}^2. \end{aligned}$$

Let  $g$  be a metric tensor field on  $M$  and  $\hat{\nabla}$  its Levi-Civita connection. Let  $(x^1, x^2)$  be an isothermal coordinate system for  $g$ , that is,  $g_{22} = \varepsilon g_{11}$ ,  $g_{11} = e^\varphi$  and  $g_{12} = 0$  for some function  $\varphi$  and  $\varepsilon = \pm$  depending on the signature of  $g$ . For an isothermal coordinate system we have

$$(20) \quad \begin{aligned} \hat{\nabla}_{\partial_1} \partial_1 &= \frac{\varphi_1}{2} \partial_1 - \varepsilon \frac{\varphi_2}{2} \partial_2, \\ \hat{\nabla}_{\partial_1} \partial_2 &= \frac{\varphi_2}{2} \partial_1 + \frac{\varphi_1}{2} \partial_2, \\ \hat{\nabla}_{\partial_2} \partial_2 &= -\varepsilon \frac{\varphi_1}{2} \partial_1 + \frac{\varphi_2}{2} \partial_2. \end{aligned}$$

The curvature  $\kappa$  of  $g$  is given by

$$(21) \quad \kappa = -\frac{\Delta \varphi}{2}.$$

Recall that for any function  $\alpha$  we have

$$(22) \quad \Delta \alpha = \frac{\alpha_{11} + \varepsilon \alpha_{22}}{e^\varphi}.$$

Assume  $\nabla$  is a connection on  $M$  such that the cubic form  $\nabla g$  is totally symmetric. The difference tensor  $K = \nabla - \hat{\nabla}$  is symmetric relative to  $g$ . For the isothermal coordinate system the symmetry is equivalent to the conditions

$$(23) \quad K_{21}^2 = \varepsilon K_{22}^1, \quad K_{12}^1 = \varepsilon K_{11}^2.$$

Assume moreover that  $\nabla \nu_g = 0$ . Then  $\text{div}^\nabla \partial_1 = \varphi_1$  and  $\text{div}^\nabla \partial_2 = \varphi_2$ . Since

$$\begin{aligned} \varphi_1 &= \text{div}^\nabla \partial_1 = \hat{\Gamma}_{11}^1 + K_{11}^1 + \hat{\Gamma}_{21}^2 + K_{21}^2, \\ \varphi_2 &= \text{div}^\nabla \partial_2 = \hat{\Gamma}_{12}^1 + K_{12}^1 + \hat{\Gamma}_{22}^2 + K_{22}^2, \end{aligned}$$

we get

$$(24) \quad K_{11}^1 = -K_{22}^1, \quad K_{11}^2 = -K_{22}^2.$$

We see that among the functions  $K_{jk}^i$ ,  $i, j, k = 1, 2$ , only two functions are independent. We choose the functions  $K_{12}^1$  and  $K_{21}^2$ . Using (17), (24) and (23), by a straightforward computation, one gets

$$(25) \quad \begin{aligned} \Lambda_{11} &= \varphi_1 K_{21}^2 - \varepsilon \varphi_2 K_{12}^1 + 2[(K_{21}^2)^2 + \varepsilon (K_{12}^1)^2], \\ \Lambda_{12} &= -\varphi_1 K_{12}^1 - \varphi_2 K_{21}^2, \\ \Lambda_{22} &= \varphi_2 K_{12}^1 - \varepsilon \varphi_1 K_{21}^2 + 2[(K_{12}^1)^2 + \varepsilon (K_{21}^2)^2]. \end{aligned}$$

We have

$$(26) \quad g(K, K) = \frac{4}{e^\varphi} [\varepsilon(K_{12}^1)^2 + (K_{21}^2)^2].$$

In the theory of affine surfaces, the function  $\frac{1}{2}g(K, K)$  is called the Pick invariant and it is usually denoted by  $J$ .

Formulas (18) for the Ricci tensor  $\text{Ric} = r_{ij}$  of the connection  $\nabla$ , after using (25), receive the following form:

$$(27) \quad \begin{aligned} (K_{21}^2)_1 &= -\frac{\varphi_{11} + \varepsilon\varphi_{22}}{2} + \varepsilon(K_{12}^1)_2 \\ &\quad - \varphi_1 K_{21}^2 + \varepsilon\varphi_2 K_{12}^1 - 2[\varepsilon(K_{12}^1)^2 + (K_{21}^2)^2] - r_{11}, \\ (K_{12}^1)_1 &= -(K_{21}^2)_2 - \varphi_1 K_{12}^1 - \varphi_2 K_{21}^2 + r_{12}, \\ (K_{21}^2)_1 &= \frac{\varphi_{11} + \varepsilon\varphi_{22}}{2} + \varepsilon(K_{12}^1)_2 \\ &\quad + \varepsilon\varphi_2 K_{12}^1 - \varphi_1 K_{21}^2 + 2\varepsilon[\varepsilon(K_{12}^1)^2 + (K_{21}^2)^2] + \varepsilon r_{22}. \end{aligned}$$

When adding and subtracting the first and the last equalities we get the following system of equalities equivalent to (27)

$$(28) \quad \begin{aligned} r_{11} + \varepsilon r_{22} &= -(\varphi_{11} + \varepsilon\varphi_{22}) - 4[\varepsilon(K_{12}^1)^2 + (K_{21}^2)^2], \\ (K_{21}^2)_1 &= \varepsilon(K_{12}^1)_2 + \varepsilon\varphi_2 K_{12}^1 - \varphi_1 K_{21}^2 + \frac{-r_{11} + \varepsilon r_{22}}{2}, \\ (K_{12}^1)_1 &= -(K_{21}^2)_2 - \varphi_1 K_{12}^1 - \varphi_2 K_{21}^2 + r_{12}. \end{aligned}$$

The first equality is the affine theorem egregium, that is, the equality

$$(29) \quad \rho = \kappa - J,$$

where  $\rho = (\text{tr}_g \text{Ric})/2$ .

#### 4. PROOF OF THEOREM 1.1

Direct proofs of necessity of the condition (2) can be found, for instance, in [2] or [3]. The proof below also gives the necessity, but we focus on the sufficiency of the condition. Assume that for a given  $g$  and some constant  $\lambda$  the equality (2) is satisfied. By the fundamental theorem we are looking for a connection  $\nabla$  such that the cubic form  $\nabla g$  is symmetric, the Ricci tensor of  $\nabla$  is equal to  $\lambda g$  and  $\nabla \nu_g = 0$ . Let  $\nabla = \hat{\nabla} + K$ . As in the previous section we will carry considerations for a fixed isothermal coordinate system for  $g$ . Instead of looking for a connection  $\nabla$  we will look for a tensor field  $K$  satisfying appropriate symmetry conditions and the system of differential equations (28), where  $\lambda = \frac{r_{11} + \varepsilon r_{22}}{2e^\varphi}$ . It will turn out that (2) is the integrability condition for the system.

Since  $g(K, K)$  should be non-zero, the tensor  $K$  should be non-zero. Suppose that  $K_{21}^2 \neq 0$ . Set

$$(30) \quad L = \varepsilon(K_{12}^1)^2 + (K_{21}^2)^2, \quad l = \frac{K_{12}^1}{K_{21}^2}.$$

The two functions  $L$  and  $l$  determine the difference tensor  $K$  up to sign. This is sufficient for our consideration because affine spheres go in pairs. More precisely, if  $g, \nabla = \hat{\nabla} + K, S = \lambda \text{id}$  constitute the induced structure on an affine sphere then  $g, \bar{\nabla} = \hat{\nabla} - K, S = \lambda \text{id}$  form the induced structure on another affine sphere. It follows from the fact that for an affine sphere  $R = \bar{R}$ , where  $R$  and  $\bar{R}$  are the curvature tensors for  $\nabla$  and  $\bar{\nabla}$ .

We now have

$$(31) \quad (K_{21}^2)^2 = \frac{L}{\varepsilon l^2 + 1}.$$

Set  $r = \lambda g$ . The system (28) now becomes

$$(32) \quad \begin{aligned} 2\frac{L}{e^\varphi} &= \kappa - \lambda, \\ (K_{21}^2)_1 &= \varepsilon(K_{12}^1)_2 + \varepsilon\varphi_2 K_{12}^1 - \varphi_1 K_{21}^2, \\ (K_{12}^1)_1 &= -(K_{21}^2)_2 - \varphi_1 K_{12}^1 - \varphi_2 K_{21}^2. \end{aligned}$$

Note that the denominator in (31) is different than 0 if and only if  $L \neq 0$ , that is, by the first equality from (32), if and only if  $\kappa - \lambda = 2L/e^\varphi \neq 0$ . Of course, we should define  $L = \frac{(\kappa-\lambda)e^\varphi}{2}$ . The function  $L$  is now given.

We want to prove that the system of the last two equations from (32) relative to unknown functions  $K_{12}^1, K_{21}^2$  has a solution. Using (30) and (32) we obtain

$$(33) \quad \begin{aligned} L_1 &= 2\{\varepsilon(K_{12}^1)_1 K_{12}^1 + (K_{21}^2)_1 K_{21}^2\} \\ &= 2\{\varepsilon[-(K_{21}^2)_2 - \varphi_1 K_{12}^1 - \varphi_2 K_{21}^2] K_{12}^1 \\ &\quad + [\varepsilon(K_{12}^1)_2 + \varepsilon\varphi_2 K_{12}^1 - \varphi_1 K_{21}^2] K_{21}^2\} \\ &= 2\{\varepsilon l_2 (K_{21}^2)^2 - \varphi_1 L\} \end{aligned}$$

and consequently

$$(34) \quad \frac{L_1}{2L} = \frac{l_2}{l^2 + \varepsilon} - \varphi_1.$$

Similarly, using (30) and (32), we get

$$(35) \quad \begin{aligned} l_1 &= \frac{(K_{12}^1)_1 K_{21}^2 - (K_{21}^2)_1 K_{12}^1}{(K_{21}^2)^2} \\ &= \frac{[-(K_{21}^2)_2 - \varphi_1 K_{12}^1 - \varphi_2 K_{21}^2] K_{21}^2 - [\varepsilon(K_{12}^1)_2 + \varepsilon\varphi_2 K_{12}^1 - \varphi_1 K_{21}^2] K_{12}^1}{(K_{21}^2)^2} \\ &= \frac{-\frac{L_2}{2} - \varphi_2 L}{(K_{21}^2)^2} \end{aligned}$$

and consequently (by (31))

$$(36) \quad \frac{l_1}{l^2 + \varepsilon} = -\varepsilon \frac{L_2}{2L} - \varepsilon\varphi_2.$$

We have got the following system of differential equations relative to  $l$

$$(37) \quad \begin{aligned} \frac{l_1}{l^2 + \varepsilon} &= -\varepsilon(\frac{1}{2} \ln |L| + \varphi)_2, \\ \frac{l_2}{l^2 + \varepsilon} &= (\frac{1}{2} \ln |L| + \varphi)_1. \end{aligned}$$

We have  $\frac{l_1}{l^2 + \varepsilon} = F_1$  and  $\frac{l_2}{l^2 + \varepsilon} = F_2$ , where  $F = \arctg l$  or  $F = \ln |\frac{l-1}{l+1}|$  depending on whether  $\varepsilon = 1$  or  $-1$ . The integrability condition for (37) is

$$-\varepsilon(\frac{1}{2} \ln |L| + \varphi)_{22} = (\frac{1}{2} \ln |L| + \varphi)_{11},$$

which is equivalent to

$$\frac{(\ln |L|)_{11} + \varepsilon(\ln |L|)_{22}}{4e^\varphi} = -\frac{\varphi_{11} + \varepsilon\varphi_{22}}{2e^\varphi},$$

that is,

$$\Delta \ln |L| = 4\kappa.$$

The last condition is equivalent to

$$(38) \quad \Delta \ln \frac{2|L|}{e^\varphi} = 6\kappa,$$

which is the desired condition (2).

Thus the system (37) has a solution  $l$  around any fixed point  $p \in M$ . As the initial condition we take  $l(p) = \beta$ , where  $\beta$  can be any real number if  $\varepsilon = 1$  and

$\beta^2 \neq 1$  if  $\varepsilon = -1$  (which again is the condition  $J \neq 0$ ). From (31) and (30) we get  $K_{12}^1$  and  $K_{21}^2$ . The other components of the tensor  $K$  are defined by using formulas (23) and (24).

We shall now check that the obtained functions  $K_{12}^1, K_{21}^2$  satisfy the last two equations of (32) if  $L = \varepsilon(K_{12}^1)^2 + (K_{21}^2)^2$ . Using also the definition of  $l$  one gets

$$(39) \quad \begin{aligned} L_1 &= 2\{\varepsilon(K_{12}^1)_1 K_{12}^1 + (K_{21}^2)_1 K_{21}^2\} \\ (K_{21}^2)^2 l_1 &= (K_{12}^1)_1 K_{21}^2 - (K_{21}^2)_1 K_{12}^1. \end{aligned}$$

We have already checked that if we substitute the quantities  $(K_{12}^1)_1, (K_{21}^2)_1$  by the right hand sides of the last two formulas of (32), then we get the equalities. We shall now regard (39) as a system of algebraic linear equations with unknowns  $(K_{12}^1)_1, (K_{21}^2)_1$ . The main determinant of (39) is equal to  $-2L \neq 0$ , hence the system of equations has only one solution. It means that the functions  $K_{12}^1, K_{21}^2$  satisfy the last two equations of (32). It follows (by (28)) that the Ricci tensor of  $\nabla = \hat{\nabla} + K$  satisfies the conditions  $\text{Ric}(\partial_1, \partial_1) = \varepsilon \text{Ric}(\partial_2, \partial_2)$  and  $\text{Ric}(\partial_1, \partial_2) = 0$ . Since the scalar curvature, say  $\rho$ , of  $\nabla$  satisfies the equality  $J + \rho = \kappa$  and we have  $J + \lambda = \kappa$ , we see that  $\text{Ric} = \lambda g$ .  $\square$

**Remark 4.1.** It is clear from the above proof that an affine sphere whose Blaschke metric is prescribed (and satisfies the condition (2)) is not unique. First of all the function  $l$  can be prescribed at a point  $p$ . Different functions  $l$  give different (non-equivalent modulo the affine special group acting on  $\mathbf{R}^3$ ) affine spheres. Of course, the difference tensors  $K$  and  $-K$  also give two different spheres if  $K \neq 0$ .

## 5. PROOF OF COROLLARIES 1.2 AND 1.3

Assume first that  $g$  satisfies (1) and  $(x^1, x^2)$  is an isothermal coordinate system for  $g$  as in the previous sections. Denote by  $\Delta_0$  the Laplacian for the coordinate system, that is,  $\Delta_0 \alpha = \alpha_{11} + \varepsilon \alpha_{22}$  for a function  $\alpha$ . This is the Laplacian for the flat metric tensor field  $g_0$  adapted to the coordinate system, that is,  $g_0(\partial_1, \partial_1) = 1$ ,  $g_0(\partial_1, \partial_2) = 0$ ,  $g_0(\partial_2, \partial_2) = \varepsilon$ , where  $\varepsilon = \pm 1$ . The equality (1) is equivalent to the equality

$$(40) \quad \Delta_0[\ln |\kappa - \lambda| + 3\varphi] = 0.$$

Hence

$$(41) \quad \ln |\kappa - \lambda| + 3\varphi = h$$

for some  $\Delta_0$ -harmonic function  $h$ . Thus  $\kappa - \lambda = \pm e^{h-3\varphi}$ . Since  $\kappa = -\frac{\Delta_0 \varphi}{2e^\varphi}$  (by (21) and (22)), the last equality is equivalent to the equality

$$(42) \quad -\frac{\Delta_0 c}{2e^c} = \mp 2 - \lambda e^{\frac{h-3c}{2}},$$

where  $c = h - 2\varphi$ . If  $\lambda = 0$  then the last equality becomes

$$(43) \quad -\frac{\Delta_0 c}{2e^c} = \pm 2.$$

It means, by (21) and (22), that  $c$  is such a function that the metric  $\bar{g} = e^c g_0$  has constant Gaussian curvature  $\pm 2$ . Note that the metric  $\bar{g}$  is defined only locally, that is, on a domain of an isothermal coordinate map.

*Proof of Corollary 1.2.* Let  $f : M \rightarrow \mathbf{R}^3$  be an improper locally strongly convex affine sphere with nowhere-vanishing Pick invariant, equivalently, with nowhere



vanishing curvature  $\kappa$  of the Blaschke metric  $g$ . Hence the above consideration can be applied. The atlas of isothermal coordinates is analytic. The Levi-Civita connection for  $\bar{g}$  is locally symmetric and therefore analytic. It follows that its curvature tensor is analytic and so is its Ricci tensor. The Ricci tensor is equal to  $\pm 2\bar{g}$ . Hence  $\bar{g}$  is analytic and  $c$  is analytic. If  $g$  is definite then a  $\bar{g}$ -harmonic function is analytic. It follows that  $\varphi = \frac{h-c}{2}$  is analytic and consequently  $g = e^{\frac{h-3c}{2}}\bar{g} = e^{\frac{h-c}{2}}g_0$  is analytic.

We have proved that the Blaschke metric  $g$  is analytic. An improper affine sphere can be locally regarded as a graph of some function and its affine normal is a constant vector. More precisely,  $f$  is, up to affine transformations of  $\mathbf{R}^3$ , given locally by

$$\mathbf{R}^2 \supset U \ni (x^1, x^2) \rightarrow (x^1, x^2, \Psi(x^1, x^2)) \in R^3,$$

where  $\Psi$  is a smooth function and its affine normal is equal to  $(0, 0, 1)$ . We have

$$g(\partial_i, \partial_j) = \Psi_{ij}.$$

Since  $g$  is analytic, so is  $\Psi$ . The proof of Corollary 1.2 is completed.  $\square$

*Proof of Corollary 1.3.* Assume now that  $h$  is an arbitrary harmonic function on some open set  $U \subset \mathbf{R}^2$  and  $c$  is such a function on  $U$  that  $g = e^c g_0$  has constant Gaussian curvature 2 or  $-2$ . Of course, such a function exists because there exist metrics of any constant Gaussian curvature. Set  $\varphi = \frac{h-c}{2}$ . By the consideration from the beginning of this section one sees that the equality (41) is satisfied for  $\kappa = -\frac{\Delta_0 \varphi}{2e^\varphi}$ .  $\square$

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